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Translated by M.D.F.

*PMM U.S.S.R.*, Vol.48, No.3, pp.347-353, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00


# ON THE STEADY MOTION OF A CRACK WITH SLIP AND SEPARATION SECTIONS ALONG THE INTERFACE OF TWO ELASTIC MATERIALS*

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The pre-Rayleigh motion of a crack (slit) with a finite slip section adjoining the edge of the crack and a semi-infinite separation section along the line connecting two elastic materials is studied under the action of a moving load. The problem is first reduced to a Hilbert boundary value problem with three different singularities for a system of two analytic functions of a complex variable. Then, by using conformal mapping techniques, analytic continuation, and elimination of singularities it is reduced to a problem with two singularities that lends itself to splitting, and consequently, of solution in Cauchy-type integrals. The length of the slip section $l$ is determined uniquely from additional physical conditions (no force of attraction on the slip section, and non-intersection of the slit edges in the separation zone) formulated in the form of inequalities. For a concentrated load at a distance $L$ from the edge

**Prikl.Matem.Mekhan.*, 48, 3, 482-489, 1984

of the slit, the solution is obtained explicitly.

It is show that $l/L \to 0$ if the elastic materials are similar in properties, and $l/L \to 1$ if the velocity of crack edge motion approaches the first of the Rayleigh velocities. When $l \ll r \ll L$ ($r$ is the distance from the edge), an intermediate asymptotic form identical with the asymptotic form of the solution of an analogous problem without taking account of the slip zone /1/ exists.

An oscillating singularity occurs on the interface of the elastic materials when the boundary conditions of the separation-adhesion type /2/ are replaced. This was apparently first found in /3/, then a number of authors examined the problem of cracks on an interface (see /4/, say).

Analysis of local solutions shows that this undesirable singularity, from the physical viewpoint, (it indicates interpenetration of the edges) can be cancelled by the introduction of a slip section ahead of the separation point, since discontinuities in the boundary conditions of the separation-slip and slip-adhesion type generate non-oscillating singularities and, moreover, unlimited compressive stresses appear on the interface behind a transverse shear crack on approaching this edge. Such a scheme (without a proper foundation) was first examined in statics for a finite crack /5/, where the problem was reduced to a singular integral equation solved numerically. For a semi-infinite crack, the ambiguity of the solution of such a problem is indicated in /6/ (actually, no condition is found for selecting a unique root of the equation to determine $l$ ). Slip sections were introduced for an analogous prupose much earlier for the impression of a stamp adhering to an elastic medium /7/.

1. A semi-infinite crack moves at a velocity $c$ over the interface of two elastic half-planes $y > 0$ (medium 1) and $y < 0$ (medium 2). We associate a Cartesian coordinate system $x = y_1$, $y = y_2$ with the slit edge (Fig.1). For $x > 0$ $(y = 0)$ the half-planes are stuck together, for $-l < x < 0$ one material slips relative to the other without friction, and on the ray $x < -l$ the slit edges do not interact (separation). Such crack motion is caused by a stationary normal and tangential load applied symmetrically to the edges in the section $x_1 < x < x_2 < -l$ (the coordinates $x_1$ and $x_2$ are given, while the quantity $l$ is to be determined). We shall study the stress $\sigma_{km}{}^j$ and velocity $u_m{}^j$ fields $(j, k, m = 1, 2;$ the superscript $j$ determines the medium) under the physical assumptions made about the crack geometry. The justification for these assumptions follows below and will include confirmation of natural additional conditions.

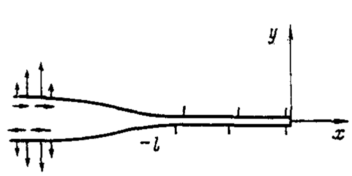

Fig.1

We will formulate the boundary conditions of the problem for $y = 0$

$$[\sigma_{2m}] = [u_m] = 0,\ x > 0;\ [\sigma_{22}] = [u_2] = 0, \sigma_{12}{}^j = 0,\ -l < x < 0 \tag{1.1}$$
$$\sigma_{22}{}^j = -\sigma(x),\ \sigma_{12}{}^j = \tau(x),\ x < -l\ \ (\sigma(x) = \tau(x) = 0$$
$$x \overline{\in} [x_1, x_2])$$

$$\sigma_{22}{}^j \leqslant 0,\ -l < x < 0,\ [U_2] = -\frac{1}{c}\int_{-l}^{x} [u_2]\, dx \geqslant 0,\quad x < -l \tag{1.2}$$

The square brackets denote a jump in the quantity on passing from medium 1 to medium 2, $j, m = 1, 2$, $(U_1, U_2)$ is the displacement vector, and $\sigma(x), \tau(x)$ are Hölder-continuous functions.

The stresses and velocities in the steady subsonic mode in the problem of the dynamic theory of elasticity (plane strain) can be expressed in terms of analytic functions $\chi_m{}^j(z_{kj})$ of the complex variable $z_{kj}$ by means of the formulas (representations close to the representations of L.A. Galin /7/)

$$\sigma_{11}{}^j = R_j^{-1}\operatorname{Re}\{-\alpha_j\beta_{2j}\chi_1{}^j(z_{1j}) + \beta_j\beta_{2j}\chi_1{}^j(z_{2j}) + \alpha_j\beta_j\chi_2{}^j(z_{1j}) - \beta_{1j}\beta_{2j}\chi_2{}^j(z_{2j})\} \tag{1.3}$$
$$\sigma_{12}{}^j = R_j^{-1}\operatorname{Im}\{\beta_{1j}\beta_{2j}\chi_1{}^j(z_{1j}) - \beta_j{}^2\chi_1{}^j(z_{2j}) - \beta_j\beta_{1j}[\chi_2{}^j(z_{1j}) - \chi_2{}^j(z_{2j})]\}$$
$$\sigma_{22}{}^j = R_j^{-1}\operatorname{Re}\{\beta_j\beta_{2j}[\chi_1{}^j(z_{1j}) - \chi_1{}^j(z_{2j})] - \beta_j{}^2\chi_2{}^j(z_{1j}) + \beta_{1j}\beta_{2j}\chi_2{}^j(z_{2j})\}$$
$$u_1{}^j = \frac{c}{2\mu_j R_j}\operatorname{Re}\{\beta_{2j}\chi_1{}^j(z_{1j}) - \beta_j\beta_{2j}\chi_1{}^j(z_{2j}) - \beta_j\chi_2{}^j(z_{1j}) + \beta_{1j}\beta_{2j}\chi_2{}^j(z_{2j})\}$$
$$u_2{}^j = \frac{c}{2\mu_j R_j}\operatorname{Im}\{-\beta_{1j}\beta_{2j}\chi_1{}^j(z_{1j}) + \beta_j\chi_1{}^j(z_{2j}) + \beta_j\beta_{1j}\chi_2{}^j(z_{1j}) - \beta_{1j}\chi_2{}^j(z_{2j})\}$$
$$\beta_{mj} = \sqrt{1 - c^2/c_{mj}^2},\ 2\beta_j = 1 + \beta_{2j}{}^2,\ \alpha_j = 1 + \beta_{1j}{}^2 - \beta_j,\ z_{mj} = x + i\beta_{mj}y$$

Here $\mu_j$ are the shear moduli, $c_{1j}$ and $c_{2j}$ are the bulk expansion and shear wave velocities, and $R_j = \beta_{1j}\beta_{2j} - \beta_j^2$ ($c_{Rj}$ are the single positive roots of the Rayleigh equation $R_j(c) = 0$).

On the interface $z_{mj} = x\ (y = 0)$ .

$$\sigma_{12}{}^j = \operatorname{Im}\chi_1{}^j,\quad \sigma_{22}{}^j = \operatorname{Re}\chi_2{}^j,\quad u_1{}^j = c\operatorname{Re}\{b_{2j}\chi_1{}^j + a_j\chi_2{}^j\} \tag{1.4}$$
$$u_2{}^j = -c\operatorname{Im}\{a_j\chi_1{}^j + b_{1j}\chi_2{}^j\},\ 2\mu_jR_j(a_j, b_{mj}) = (\beta_{1j}\beta_{2j} - \beta_j, \beta_{mj}(1-\beta_j))$$

We will seek a solution in the energy class of functions with everywhere finite displacements. Hence, the following estimates result ($z = x + iy$ is an auxiliary variable):

$$|\chi_m{}^j| < \frac{\text{const}}{|z - z_k|^{1/2}},\quad z\to z_k,\ |\chi_m{}^j| < \frac{\text{const}}{|z^{1+\varepsilon}|}\quad z\to\infty \tag{1.5}$$
$$\varepsilon > 0,\quad z_1 = 0,\quad z_2 = -l\quad (k, j, m = 1, 2)$$

The continuity condition for the stresses $\sigma_{m2}{}^j$ on the whole axis $y = 0$, that results from (1.1) is equivalent, taking account of (1.4) and (1.5), to the equations

$$\chi_1{}^1(z) = -\overline{\chi_1{}^2}(\bar z) \equiv \chi_1(z), \chi_2{}^1(z) = \overline{\chi_2{}^2(\bar z)} \equiv \chi_2(z), \operatorname{Im} z > 0$$

which reduce the number of unknown complex functions to two, while the expressions for the velocity jumps take the form $(y = 0)$

$$[u_1] = c\operatorname{Re}\{q\chi_1 + d\chi_2\},\ [u_2] = -c\operatorname{Im}\{d\chi_1 + p\chi_2\} \tag{1.6}$$
$$d = a_1 - a_2,\ p = b_{11} + b_{12},\ q = b_{21} + b_{22}$$

The remaining conditions (1.1) in terms of the $\chi_m$ are boundary conditions of a Riemann-Hilbert problem /8/

$$\operatorname{Im}\chi_1 = \tau(x),\ \operatorname{Re}\chi_2 = -\sigma(x),\ x < -l,\ \operatorname{Im}\chi_1 = \operatorname{Im}\chi_2 = 0, \tag{1.7}$$
$$-l < x < 0$$
$$q\operatorname{Re}\chi_1 + d\operatorname{Re}\chi_2 = 0,\ d\operatorname{Im}\chi_1 + p\operatorname{Im}\chi_2 = 0,\ x > 0$$

It can be rewritten in the form of a Hilbert problem /9/ if a piecewise-holomorphic vector is introduced

$$\boldsymbol{\chi} = (\chi_1, \chi_2),\ \operatorname{Im} z > 0;\ \boldsymbol{\chi}(z) = (\overline{\chi_1(\bar z)}, \overline{\chi_2(\bar z)}),\ \operatorname{Im} z < 0 \tag{1.8}$$
$$\boldsymbol{\chi}^+ = g(x)\boldsymbol{\chi}^- + G(x),\quad G = 2(i\tau(x), -\sigma(x))$$
$$g_0 = \begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix},\quad x < -l;\quad g_1 = \begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix},\quad -l < x < 0$$
$$g_2 = \frac{1}{S}\begin{Vmatrix}d^2 + pq & 2pd\\ -2qd & -d^2 - pq\end{Vmatrix}$$
$$x > 0\ (S = d^2 - pq)$$

The superscripts plus or minus denote narrowing on the real axis from above or below. Because of (1.7), the vector $\boldsymbol{\chi}$ is analytic in a plane cut along the real axis $x < -l$ and $x > 0$. It satisfies the conjugate conditions (1.8) with constraints on the behaviour at the singularities (1.5). This generalized problem with three singular points is among the singular problems of a linear conjugate since the location of one singular point is not known in advance. Such problems occur in the theory of filtration where a general algorithm for finding the effective solution is developed /10/. It consists of constructing expansions of functions (canonical local solutions firstly) in Laurent series in the neighbourhood of each of the singularities and relying substantially on the analytic theory of differential equations.

2. Following /9, 10/, we analyse the local solutions. We let $\lambda_{kn}$ ($k = 1, 2$; $n = 1, 2, 3$) denote the roots of the characteristic equations for the singularities

$$\det\|g_{k-1}g_k^{-1} - \lambda E\| = 0,\quad \det\|g_0g_2^{-1} - \lambda E\| = 0 \tag{2.1}$$

where $E$ is the unit matrix, and $\lambda$ is a parameter. For problem (1.8), these roots are the following:

$$\lambda_{11} = \lambda_{12} = 1,\quad \lambda_{22} = \lambda_{21} = -1,\quad \lambda_{13} = \lambda_{23}{}^{-1} = (d + \sqrt{pq})/(d - \sqrt{pq}).$$

For $c < \min(c_{R1}, c_{R2}) \equiv c_*$ we have $p > 0$, $q > 0$, $S < 0$, $\sqrt{pq} > |d|$, hence $-\infty < \lambda_{23} < \lambda_{13} < 0$. Furthermore, to be specific we consider $d < 0$ (if $d > 0$, then medium 1 and medium 2 should be interchanged; the case $d = 0$ will be examined separately). We introduce the indices $\rho_{kn} = (\ln\lambda_{kn})/(2\pi i)$, defined apart from integers. Because of (1.5), all the singular points are regular, there are no cases of multiple roots of (2.1), and then the principal parts of the Laurent series expansions of the canonical solutions $\chi_{kn}$ around the singular points are proportional to $(z - z_m)^{\rho_{km}}$ ($k, m, n = 1, 2$) and $z^{-\rho_{k3}}$, respectively /9, 10/. The functions $\chi_m$ can be represented in the form of linear combinations of pairs of functions $\chi_{kn}$ around each singularity $z = z_n$. We select the indices themselves by starting from the estimates (1.5)

$$\rho_{11} = \rho_{12} = 0, \ \ \rho_{21} = \rho_{22} = -{}^1/_2, \ \ \rho_{13} = \bar{\rho}_{23} = {}^3/_2 - i\alpha \qquad (2.2)$$
$$2\pi\alpha = \ln\lambda, \ \lambda = (\sqrt{pq} - d)/(\sqrt{pq} + d)$$

The branch of the solution is thereby determined.

3. The conformal mapping

$$\omega = (2z + l + 2\sqrt{z(z+l)})/l \ \ (z = (l/4)(\omega + \omega^{-1} - 2)) \qquad (3.1)$$
$$\sqrt{z(z+l)}^+ = -\sqrt{z(z+l)}^-, \ x \,\overline{\in}\, [-l, \ 0], \ \sqrt{z(z+l)}^+ = \sqrt{z(z+l)}^-, x \in ]-l, 0[$$

transfers the z plane with the cuts indicated in Sect.1 inot the upper $\omega$ half-plane. The points $\omega = -1, 1$ correspond to the points $z = -l, 0$, and the segment $\{-l \leqslant x \leqslant 0, y = 0\}$ transfer into a unit semicircle in the $\omega$ half-plane

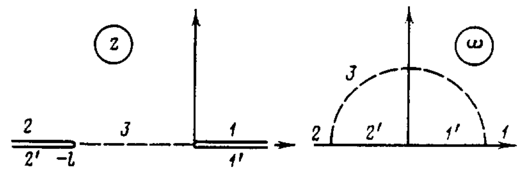


Fig.2

To eliminate the singularities (poles) at the points $\omega = \pm 1$, we introduce the new unknown functions

$$Y_1 = \sqrt{z(\omega)}(\omega + 1)\,\chi_1, \ \ iY_2 = \sqrt{z(\omega)}(\omega + 1)\,\chi_2 \qquad (3.2)$$
$$\sqrt{z}^+ = -\sqrt{z}^-, \ x > 0, \ \ \sqrt{z}^+ = \sqrt{z}^-, \ x < 0$$

Because of (2.2) and (3.1), the points $\omega = \pm 1$ will be ordinary points for the vector functions $\mathbf{Y} = (Y_1, Y_2)$. Furthermore, we again consider the vector $\mathbf{Y}$ piecewise-holomorphic, given in the whole $\omega$ plane with a cut along the real axis $(Y_k(z) = -\overline{Y_k(\bar{z})}, \ \operatorname{Im} z < 0)$. The boundary values of $\mathbf{Y}$ satisfy the following conjugate conditions $(\omega = \xi + i\eta)$:

$$\mathbf{Y}^+ = D\mathbf{Y}^-, \ \ \xi > 0, \ \ \mathbf{Y}^+ = \mathbf{Y}^- + 2i\mathbf{\Sigma}, \ \ \xi < 0 \qquad (3.3)$$
$$\frac{\mathbf{\Sigma}}{i\sqrt{|z(\xi)|}\,(\xi+1)} = \begin{cases} (\tau, \sigma), & -\infty < \xi < -1 \\ (-\tau, \sigma), & -1 < \xi < 0 \end{cases}$$
$$D = \frac{-1}{S}\begin{Vmatrix} d^2 + pq & -2dpi \\ 2dqi & d^2 + pq \end{Vmatrix}$$

Problem (3.3) is already a problem with two singularities $\omega = 0, \infty$.

The matrix $T = \{t_{km}\}, \ t_{11} = t_{12} = 1, \ t_{21} = \bar{t}_{22} = i\sqrt{q/p}$ diagonalizes the matrix $D$

$$\Lambda = T^{-1}DT = \begin{Vmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{Vmatrix} \ \ (\lambda > 1)$$

From (3.3) we obtain a split conjugate problem whose index is zero /8/ for the new vector-function $\mathbf{W} = T^{-1}\mathbf{Y} \equiv (W_1, W_2)$:

$$\mathbf{W}^+ = \Lambda\mathbf{W}^-, \ \xi > 0, \ \mathbf{W}^+ = \mathbf{W}^- + 2iT^{-1}\mathbf{\Sigma}, \ \xi < 0 \qquad (3.4)$$

The single solution of (3.4) satisfying the necessary condition at infinity $(\mathbf{W} = O(1), \ \omega \to \infty)$ can be written in the form /8/

$$W_k = \frac{\omega^{\rho_k}}{\pi}\int_{-\infty}^{0}\frac{W_k^\circ(\xi)\,d\xi}{\xi^{\rho_k}(\xi - \omega)}, \ \ \rho_1 = \bar{\rho}_2 = 1 - i\alpha, \ \ (W_1^\circ, W_2^\circ) = T^{-1}\mathbf{\Sigma}$$
$$(\omega^{i\alpha} = \exp(i\alpha\ln|\omega| - \alpha\arg\omega), \ 0 \leqslant \arg\omega \leqslant 2\pi)$$

The vector $\boldsymbol{\chi}$ is expressed in terms of $\mathbf{W}$ by means of the formula

$$\boldsymbol{\chi} = \begin{Vmatrix} 1 & 0 \\ 0 & i \end{Vmatrix}\frac{T\mathbf{W}}{\sqrt{z}\,(\omega+1)}$$

As a result of calculating the product of the matrices and vectors and transforming the integral over the segment $-1 \leqslant \xi \leqslant 0$ to an integral over the ray $-\infty < \xi \leqslant -1$ using the symmetry properties of the functions, we obtain the main result in the form

$$\chi = \frac{i}{2\pi\sqrt{z}} \int_{\xi_1}^{\xi_2} \left[ \mathbf{H}^{(1)} + i\frac{(\omega-1)(\xi+1)}{(\omega+1)(\xi-1)} \mathbf{H}^{(2)} \right] \frac{(\xi-\xi^{-1})\sqrt{|x(\xi)|}}{(\xi-\omega)(\xi-\omega^{-1})}\, d\xi \tag{3.5}$$

$$\mathbf{H}^{(1)} = [\sigma\sqrt{p/q}\sin(\alpha\ln|\xi|) + \tau\cos(\alpha\ln|\xi|)] \quad (h_+,\ h_-\sqrt{q/p})$$

$$\mathbf{H}^{(2)} = [\sigma\sqrt{p/q}\cos(\alpha\ln|\xi| - \tau\sin(\alpha\ln|\xi|)] \quad (h_-,\ h_+\sqrt{q/p})$$

$$h_\pm = \lambda^{1/2}\omega^{i\alpha} \pm \lambda^{-1/2}\omega^{-i\alpha},\quad \omega = \omega(z),\quad \xi_k = \xi(x_k) \quad (k = 1,\ 2)$$

The angular distribution of the desired functions around the point $z = 0$ and the intensity coefficients can be obtained from the asymptotic forms of this solution as $z \to 0$ independently of the parameter $l$ (see also /11/)

$$\chi \sim \frac{-i}{2\pi\sqrt{\lambda z}} \left\| \begin{matrix} \lambda+1 \\ \lambda-1 \end{matrix} \right\| \int_{x_1}^{x_2} \left[ \sigma\sqrt{\frac{p}{q}}\sin(\alpha\ln|\xi(x)|) + \tau\cos(\alpha\ln|\xi(x)|) \right] \frac{dx}{\sqrt{|x|}}$$

Analysis of the asymptotic forms of the solution in the neighbourhood of the point $z = -l$ shows that that combination of symbols $\sigma_{22}(x, 0)$ as $x \to -l + 0$ and $[u_2]$ as $x \to -l - 0$ corresponds to the singularity of the solution $\sim (z + l)^{-1/2}$ irrespective of the sign of the coefficient, so that it is impossible simultaneously to satisfy (locally) the inequalities (1.2). Hence, the deduction about the necessity to extinguish the contribution of the pole $\omega = -1$ to the solution follows. Equating the residue of the function $\mathbf{X}(\omega)$ at the point $\omega = -1$ to zero, we obtain an equation to determine $l$ (it is identical for each of the vector components)

$$\int_{x_1}^{x_2} [\sigma(x)\sqrt{p/q}\cos(\alpha\ln|\xi|) - \tau(x)\sin(\alpha\ln|\xi|)]\sqrt{|x|}\,\xi^{-1}\xi'\,dx = 0$$

$$l\xi = 2x + l - 2\sqrt{x(x+l)},\ \xi' = d\xi/dx = l^{-1}[2 + (1 + l/x)^{1/2} + (1 + l/x)^{-1/2}]$$

The solution of this equation is not unique. Global confirmation of conditions (1.2) enables the unique root to be extracted.

The subsequent reasoning is for the case of a concentrated load $\sigma = \Sigma\delta(x + L)$, $\tau = T\delta(x + L)$. The equation for $l$ takes the form

$$\sqrt{p/q}\,\Sigma\cos(\alpha\ln|\xi_L|) - \mathrm{T}\sin(\alpha\ln|\xi_L|) = 0 \quad (\xi_L = \xi(L)) \tag{3.6}$$

Taking account of (3.6) and the assumption made regarding the sign of $d$, we obtain from the condition $\sigma_{22} \leqslant 0$ and $\omega = e^{i\theta}$

$$\mathrm{T}\cos(\alpha\ln|\xi_L|) + \sqrt{p/q}\,\Sigma\sin(\alpha\ln|\xi_L|) = \sqrt{\mathrm{T}^2 + (p/q)\Sigma^2} \tag{3.7}$$

$$\sin(\alpha\ln|\xi_L|) > 0,\quad \Sigma > 0$$

Conversely, from (3.7) it follows that $\sigma_{22} \leqslant 0$ for $\omega = e^{i\theta}$ $(0 < \theta < \pi)$.

Taking (3.6) and (3.7) into account as well as the equation $\xi'(L) = 4\xi_L^2/[l(\xi_L^2 - 1)]$ we can write the final form of the solution (3.5) for the concentrated load

$$\chi = \frac{2i\xi_L\sqrt{L(q\mathrm{T}^2 + p\Sigma^2)}}{\pi l\sqrt{\lambda z}(\xi_L - \omega)(\xi_L - \omega^{-1})} \left\| \begin{matrix} q^{-1/2}(\omega^{i\alpha}\lambda + \omega^{-i\alpha}) \\ p^{-1/2}(\omega^{i\alpha}\lambda - \omega^{-i\alpha}) \end{matrix} \right\| \tag{3.8}$$

The condition of non-intersection of the cut edges remains unverified (although it is satisfied only asymptotically as $x \to -l - 0$). On the basis of (1.2), (1.6) and (3.8), we can write

$$[U_2] = -d\mathrm{T}h(-x - L) - 2I\sqrt{\frac{L(q\mathrm{T}^2 + p\Sigma^2)}{lp}}$$

$$I(\xi_0) = \frac{1}{\pi}\int_{-1}^{\xi_0} \frac{\sin(\alpha\ln|\xi|)(\xi+1)\,d\xi}{|\xi|^{3/2}(\xi_L - \xi)(\xi_L - \xi^{-1})}$$

where $h(x)$ is a step function. It can be shown that for $0 \leqslant \alpha\ln|\xi_L| \leqslant \pi$ the function $I(\xi)$ does not change sign, while $[U_2] > 0$ within the interval $\xi_L < \xi < -1$ $(-L < x < -l)$. For other (positive because $|\xi_L| > 1$) values of $\alpha\ln|\xi_L|$, the root of (3.6), the function $I(\xi)$ is sign-variable, i.e., the condition of "non-intersection" is disturbed in the domain $l < -x < L$. Hence, (while taking (3.7) into account $\mathrm{T} \neq 0$ for $\Sigma = 0$ ), the selection of the single root of (3.6) follows

$$|\xi_L| = e^{\gamma/\alpha},\quad l = 4L|\xi_L|/(|\xi_L| + 1)^2$$

$$\gamma = \min\left\{ \operatorname{Arctg}\left( \frac{\Sigma}{\mathrm{T}}\sqrt{\frac{p}{q}} \right) > 0 \right\} \quad (\Sigma \neq 0)$$

$$\gamma = \frac{\pi}{2} \quad (\mathrm{T} = 0, \Sigma \neq 0)$$
$$\gamma = 0,\ l = L\ (\Sigma = 0,\ \mathrm{T} > 0),\ \gamma = \pi\ (\Sigma = 0,\ \mathrm{T} < 0)$$

At the point $x = -L$ the function $[U_2](x)$ has a logarithmic singularity, with the exception of the case $\Sigma = 0, \mathrm{T} > 0$ when it experiences a discontinuity of the first kind. The requirement $[U_2] \geqslant 0$ is apparently not satisfied completely successfully for $x < -L$, neither for any combinations of $\Sigma$ and $T$: it oscillates as $x \to -\infty$, damps out and is sign-varying (for $\mathrm{T} = 0, \Sigma > 0$, and $\Sigma = 0, \mathrm{T} \neq 0$ this can be proved). The first minimum (in the negative domain) is reached at the point $-x = L_* = l\,(e^{-\pi/\alpha} + 1)/4$, with the exception of the case $\mathrm{T} < 0, \Sigma = 0$ in which the function $[U_2]$ has a logarithmic singularity with a change in sign on passing through the point $x = -L$. However, the solution should be considered acceptable in cases when $L_* \gg L$ since the repulsion forces not taken into account in the formulation and which occur during interaction of the edges will be insignificant in magnitude in the first place, and located far from the edge of the slit in the second (a computation for $l/L = 0.5$ in the typical cases $\Sigma > 0, \mathrm{T} = 0$ and $\mathrm{T} > 0, \Sigma = 0$ yields the value $L_* \approx 4.5\,L$). By introducing a certain additional system of small forces acting on the cut edges for $x \ll -L$, it can be hoped to satisfy the "non-intersection" condition completely in these cases.

4. In a small neighbourhood of the cut edge ($|z| \ll l$) the behaviour of the solution on the cut is the following:

$$\begin{aligned} &\sigma_{12}{}^j \sim K_2\,(2\pi x)^{-1/2} \sim \mathrm{const}\cdot u_2{}^j,\ \sigma_{22}{}^j \sim u_1{}^j = O\,(1),\ x \to +0 \\ &\sigma_{22}{}^j \sim K_1\,(2\pi\,|\,x\,|)^{-1/2} \sim d\,[u_1]/(cS),\ x \to -0 \\ &K_2 = -(1 + \lambda)[(p\Sigma^2 + q\mathrm{T}^2)/(2\pi\lambda qL)]^{1/2} = -(p/d)\,K_1 \end{aligned} \tag{4.1}$$

As $c \to 0$ (statics) these results, which are independent of $l$, are in agreement with the solution /6/ for the scheme /5/, where the expressions for the static intensity coefficients $K_1$ and $K_2$ are exact, while not asymptotically exact as $l/L \to 0$, as is indicated in /6/.

Following /4/, we calculate the energy flux $F$ per unit length of the cut edge: $F = -cSK_2{}^2/(4p)$ (as is clarified, the singularity in the stress $\sigma^j{}_{22}$ to the left of the cut edge induces no contribution to the quantity $F$). It equals the power developable by the load; there is no energy sink at the other singular points. The slip section remains plane since it follows from (1.4) and (1.7) that $u_2{}^j \equiv 0$. Near the separation point

$$\sigma_{22}{}^j(x,\ 0) = O\,((l + x)^{1/2}), \quad x \to -l + 0 \tag{4.2}$$

If $l \ll L$, and moreover, a $z$ can be indicated such that $l \ll |\,z\,| \ll L$, then the intermediate asymptotic form holds. On the continuation of the crack for $l \ll x \ll L$ ($1 \ll \xi \ll \xi_L \approx 4L/l$)

$$\sigma_{22}{}^j + i\sigma_{12}{}^j \sim \frac{K_2}{\sqrt{2\pi x}}\left[\sqrt{\frac{q}{p}}\sin\left(\alpha\ln\frac{4x}{l}\right) - i\cos\left(\alpha\ln\frac{4x}{l}\right)\right] \tag{4.3}$$

For $\mathrm{T} = 0$ we have $\alpha\ln\,(4L/l) \approx \alpha\ln\,|\,\xi_L\,| = \pi/2$, and (4.3) agrees with the corresponding asymptotic form of the solution of the problem of a crack without taking account of the slip zone /1/. Passing to the limit of the case of identical materials ($\mu_1 \to \mu_2,\ c_{m1} \to c_{m2}$), we obtain $\alpha \to 0,\ l/L \to 0\ (\Sigma > 0)$. The domains in which the local asymptotic forms (4.1) and (4.2) act hence vanish while the intermediate asymptotic form goes over into the asymptotic form of the corresponding problem about a separation crack (if $\Sigma = 0$, then it is shear) in a homogeneous plane. On the other hand, by allowing the medium 2 to tend to a rigid medium (and the parameters $\mu_2, c_{12}, c_{22}$ to tend to infinity) for $c = \mathrm{const}$, we find that the quantity $l/L$ grows and reaches a maximum. This fact is in agreement with the following physical explanation of the appearance of a contact area with slip: materials experience different tension of the outer layers during bending, consequently, tangential forces also appear on the interface during the attempt to separate two elastic materials from each other,

As the velocity of the edge approaches the Rayleigh value ($c \to c_*$) it can be seen that $\lambda \to \infty,\ l \to L$. Note that the quantities $l/L$ are of the order of $5 \times 10^{-4}$ for $\mathrm{T} = 0$ in statics. If $c = c_d$, where $c_d$ is a root of the equation $d\,(c) = 0$, then $\lambda = 1,\ \alpha = 0,\ l = 0$, as in the case of identical materials (the behaviour of the solutions is also qualitatively identical). The root $c_d$ can exist in the velocity range being studied if $\mu_2\,(1 - 2\nu_1) < \mu_1\,(1 - 2\nu_2)$ and $c_* = c_{R1}$ or $\mu_2\,(1 - 2\nu_1) > \mu_1\,(1 - 2\nu_2)$ and $c_* = c_{R2}$ ($\nu_j$ are Poisson's ratios).

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Translated by M.D.F.

*PMM U.S.S.R.*, Vol.48, No.3, pp.353-356, 1984
Printed in Great Britain

0021-8928/84 $10.00+0.00


# ASYMPTOTICALLY PENDULUM-LIKE MOTIONS OF THE HESS-APPEL'ROT GYROSCOPE*

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Asymptotically pendulum-like motions of a heavy rigid body whose centre of gravity lies on the perpendicular to the circular cross-section of the gyration ellipsoid (the Hess-Appel'rot gyroscope) is investigated. Lyapunov's theorem is used to show that the initial position and initial angular velocity of this gyroscope can also be chosen such, that its motion will tend asymptotically, as time increases without limit, to rotation about the horizontal axis. Since in this case the initial conditions do not satisfy the invariant Hess relation, it follows that the results described cannot be obtained by direct generalisation of /1/ where the asymptotically pendulum-like motions were obtained for the special case of the Hess solution by constructing the phase trajectories.

Various examples of asymptotic motions in the classical problem of the motion of a heavy rigid body with a fixed point are shown in /1—5/.

Let the centre of gravity of a heavy rigid body with a fixed point lie on the perpendicular to the circular cross-section of the gyration ellipsoid constructed at the fixed point. We attach to this body a special coordinate system, and write its equations of motion about the fixed point in dimensionless coordinates /6/

$$\begin{aligned}
&x^{\cdot} = -xz, \quad y^{\cdot} = (a - a_2)xz + yz - \nu_2 \\
&z^{\cdot} = -(a - a_2)xy + x^2 - y^2 + \nu_1 \\
&\nu^{\cdot} = \omega_2\nu_1 - \omega_1\nu_2, \quad \nu_1^{\cdot} = \omega\nu_2 - \omega_2\nu, \quad \nu_2^{\cdot} = \omega_1\nu - \omega\nu_1 \\
&\omega = ax + y, \quad \omega_1 = a_2y + x, \quad \omega_2 = a_2z
\end{aligned} \tag{1}$$

where $x, y, z$ are the components of the angular momentum vector, $\omega, \omega_1, \omega_2$ are the components of the angular velocity vector, $\nu, \nu_1, \nu_2$ are the components of the unit vector indicating the direction of the force of gravity, $a, a_2$ are the dimensionless parameters characterising the ratios of the gyration tensor components, and a dot accompanying the variable denotes differentiation with respect to time.

Equations (1) have the following first integrals:

$$\begin{aligned}
&ax^2 + a_2(y^2 + z^2) + 2xy - 2\nu = 2E \\
&\nu^2 + \nu_1^2 + \nu_2^2 = 1, \quad x\nu + y\nu_1 + z\nu_2 = k
\end{aligned} \tag{2}$$

We shall use the same variables accompanied by an asterisk to write the particular solution of (1), describing the motion of a body about the horizontal axis, and the values of the constants of the integrals (2) of this solution. Then the solution will have the form

**Prikl.Matem.Mekhan.*, 48, 3, 490-493, 1984